Distributed coverage while not being covered

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Abstract— **We consider the problem of cooperatively covering a group of static targets while simultaneously minimizing exposure from a different set of static locations. Starting from the work of Bullo et al. [8] who formalized the problem of cooperative sensor coverage as a multicenter optimization problem, we show that also this problem can be formulated and solved using related concepts. However, we evidence that the resulting function to be optimized loses some desirable properties and requires a more sophisticated controller. After having identified the peculiar aspects of this new function to be optimized and having studied its critical points, we provide a controller and show that it will drive the system to a stable local optimizer. The control strategy is distributed in the sense of Voronoi, and it implements a distributed version of the gradient projection method known in the optimization literature. Numerical simulations illustrate how the controller works, and we conclude sketching some theoretical properties.**

I. Introduction

The use of teams of cooperative mobile robots for tasks involving surveillance and search continues to be a major driver for research in multi-robot systems. Surveillance is here intended as the activity of continuous information gathering (sensing) of areas of interest. Information gathered by the robots is then passed to an inference system (or human operator) making decisions based on the evidence collected. A typical example consists in monitoring the entrances of an area to prevent access by unauthorized subjects. Starting from the seminal work by Bullo and co-workers [8], there has been a flurry of results based on so-called multicenter functions. In essence, the question is where a team of robots should be deployed in order to provide optimal coverage of a set of predetermined locations. The importance of these locations is measured by a *density function* indicating which locations are more relevant. A performance index is used to measure the quality of a given deployment. This quality is then influenced by the robots' locations and the density function, but also by the sensing capabilities of the robots. Typically, the density function indicates the likelihood that a certain event will occur at a given location, whereas sensing quality determines how well a robot will cover a given point from its current position. Besides the optimization aspect there is a distributed control facet, i.e. a distributed control law robustly driving the team to a desired final arrangement is needed.

In this paper we consider a related problem where the robot team is asked to provide coverage to certain locations while simultaneously staying away from *undesirable* sites. This scenario occurs in numerous situations. Undesirable locations may be congested areas the robots should avoid in order not to interfere with other agents, or areas characterized by environmental features that may impair the robots themselves, like excessive heat, or areas in proximity to electromagnetic sources that may jam the robots' communication equipment. If one assumes that hostile subjects are placed at the locations to be avoided, the task may be thought as the problem of simultaneously covering interesting locations while not being covered by the hostile subjects – hence the title of this contribution. We address this problem building upon the aforementioned multicenter optimization framework. The multicenter approach is attractive because it allows us to formally define and reason about simple control laws that have immediate meaning. However, its formal setting leads to principled design and analysis, as opposed to heuristic methods often used for this class of problems, e.g. when using behavior-based controllers. Moreover, by building upon this line of work we benefit from a significant body of recently developed results in this area. Nevertheless, we show that even though the problems are formulated using similar concepts, the functions to be optimized have structurally different properties. Not surprisingly, the function we consider in this paper is more complicated, because it aims at solving a problem which is more general than coverage. Indeed it is easily seen that coverage problems are a simplified instance of the questions we ask in this paper. However, by carefully analyzing the analytic structure of the function we propose, it is possible to identify its peculiar aspects and devise an appropriate distributed control law that enjoys most of the properties possessed by formerly developed algorithms for cooperative coverage.

The contributions of this paper are the following:

- the problem of covert observation is formalized as a multicenter optimization problem;
- we provide a thorough study of the function to be optimized, outlining fundamental differences with those used in coverage-only problems. Differences emerge in the nature of the function's critical points, and in the gradient descent control law.
- we provide a control policy that is guaranteed to converge towards a local optimum and is distributed in the sense of Voronoi.

The above analytical findings are complemented by a section showing some examples of produced paths and performances.

This paper is organized as follows. Related work is briefly summarized in Section II. Section III provides the mathemat-

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ical formulation of the problem and introduces the function to be optimized. Next, in Section IV we study the critical points of the function and identify necessary conditions for optimality. A distributed controller is then presented in Section V, and simulation results are shown in Section VI. Theoretical properties are summarized in Section VII, and finally in Section VIII we identify a number of directions for further study and we summarize the results presented.

II. Related Work

Coverage problems have been extensively studied and a systematic discussion goes beyond the scope of this manuscript. One may identify two main threads. Behavior-based approaches have been proposed and practically implemented on real robots. Parker's Cooperative Multirobot Observation of Multiple Moving Targets (CMOMMT) [14] shows how simple rules may lead to a rich group behavior attaining a complex task like observing multiple moving targets. The drawback of behavior-based systems is in their inherently heuristic nature, and in the impossibility to derive quantitative conclusions. At the other side of the spectrum are approaches developed in the control theory community, more sophisticated from a mathematical point of view, but less frequently implemented and tested on robotic hardware (see however [16] for a practical implementation of the methods discussed in the following). Bullo, Cortéz, and Martinez have proposed to treat this problem as a multicenter optimization routine executed by a distributed robot team. Their recent book [5] provides a comprehensive guide to the topic, and the next section offers a formal mathematical introduction borrowed from their work. Starting from [8] various extensions were considered, like the case of robots where the density function is incrementally discovered [12], or the case of heterogenous teams with different sensors [15]. Breitenmoser et al. [4] study the problem of coverage in non-convex environment. In this situation it is possible that robots will be *pushed* outside the environment while trying to move according to their control law – an aspect characterizing also the problem we study in this paper. However, in [4] this aspect is handled by an application of a bug algorithm, whereas we solve it using tools from optimization theory. Schwager et al. propose a general treatment of deployment methods based on potential field approaches [17]. We notice however that their framework does not cover the case we study in this paper because, as it will be evidenced in the next section, the performance function we optimize is defined in terms of a service function that can be both positive and negative, while they require strict positiveness.

The problem of completing a task while not being observed has been investigated less extensively, and is sometimes known in literature as covert robotics or stealth observation, where *covertness* is the term used to indicate a robot does not fall within the sensing range of an hostile entity. One of our former contributions in this area is formalized within the CMOMMT framework and proposes coordinated actions to let multiple robots escape multiple mobile observers [11]. The paper however considers covertness only, i.e. it does not require to concurrently perform a coverage assignment. Sukhatme and colleagues have also investigated this area, starting from the problem of a single robot remaining hidden while reaching an assigned location [3], then considering multiple evaders and a single static observer [19], and finally considering dynamic environments [18]. Adopting a behavior-based design, these papers do not provide formal performance bounds or analytic criteria establishing convergence. Similar ideas, although defined on a discretized environment, were proposed by Marzouqi and Jarvis [13]. Hsu et al [2] considered the problem of covert approaching, i.e. tracking a moving object without falling within its sensing range. The authors do not assume prior knowledge of the environment or of the target motion, and propose a greedy strategy. The problem of performing a cooperative coverage task while avoiding exposure to multiple observers appears therefore to be novel, and the use of multicenter functions has not been explored in the area of covert robotics.

III. Mathematical Foundations

A. Notation

We first introduce some notation used in this manuscript and we embrace the notation presented in [5]. Let $\mathbb{R}$ be the set of real numbers, $\mathbb{R}^+$ be the set of positive real numbers, and $\mathbb{R}^+_0$ be the set of non negative real numbers. Let $x_1, \ldots, x_n$ be n points in $\mathbb{R}^2$, and let x_i^1, x_i^2 be the components of x_i. We say that the $2n$ dimensional vector $(x_1, \ldots, x_n)$ is a *non singular* configuration if $i \neq j \Rightarrow x_i \neq x_j$, i.e. there are no repeated points. Given a set $S \subset \mathbb{R}^2$ of strictly positive measure and a function $\phi : \mathbb{R}^2 \to \mathbb{R}^+$, the generalized area and center of mass of S are, respectively

$$\mathrm{A}_\phi(S) = \int_S \phi(q)dq \qquad \mathrm{CM}_\phi(S) = \frac{1}{\mathrm{A}_\phi(S)} \int_S q\phi(q)dq.$$

Note that since ϕ is defined to be positive in S, then $\mathrm{A}_\phi(S)$ is always postitive and then $\mathrm{CM}_\phi(S)$ is well-defined. The reader should also note that despite ϕ is often referred to as "density", it is not a probability density function, so no normalization is required. Let $(x_1, \ldots, x_n)$ be a non singular configuration. Its points induce a *Voronoi* subdivision of the plane, i.e. every point x_i is associated with the set $V(x_i)$ defined as follows [9]:

$$V(x_i) = \{ \ p \in \mathbb{R}^2 \mid ||p - x_i||_2 \leq ||p - x_j||_2 \ \forall j \neq i \ \}.$$

In the following $\mathcal{E} \subset \mathbb{R}^2$ will indicate a convex planar region with polygonal boundary. The planar region bounded by $\mathcal{E}$ is given by the intersection of m halfplanes defined by inequalities like the following:

$$a_1 x_1 + a_2 x_2 \leq b.$$

where $(x_1, x_2) \in \mathbb{R}^2$. Given a function $\mathcal{G} : \mathbb{R}^k \to \mathbb{R}$, we indicate with $\nabla\mathcal{G}$ its gradient. Moreover, we will use $\nabla_i\mathcal{G}$ to designate individual components of the gradient.

B. Function to be optimized

Robots are requested to cover certain locations while avoiding to be too close to certain other regions. This requirement is formalized introducing two functions $\phi_i : \mathcal{E} \to \mathbb{R}^+$, with $i \in \{1, 2\}$. In the following ϕ_1 identifies the regions to cover, and ϕ_2 the areas to stay away from. Next, we introduce a *performance function* $f : \mathbb{R}_0^+ \to \mathbb{R}$. The performance function is used to model the intensity of interaction occurring between a robot and a site of either kind. We assume that f is continuous, differentiable, and non-increasing. In other words, the closer the robot to a site, the stronger the interaction, either positive (with sites of the first type) or negative (with sites of the second type). The intuition behind these hypotheses is that the farther a robot is to a location, the weaker the observation is possible, either from the robot to the area to be covered, or from the hostile observer to the robot.

While in general one may envision different performance functions for sites of interest and sites to be avoided, this complication is here circumvented, and differences will be accounted for by introducing suitable scaling factors k_1 and k_2 (see next equation). We now introduce the multicenter function that is the main topic of this paper:

$$\mathcal{H}(x_1, \ldots, x_n) = \int_{\mathcal{E}} \left[k_1 \max_{\substack{x_i \in \mathcal{E} \\ i \in \{1 \ldots n\}}} f(||q - x_i||_2)\phi_1(q) - k_2 \max_{\substack{x_i \in \mathcal{E} \\ i \in \{1 \ldots n\}}} f(||q - x_i||_2)\phi_2(q) \right] dq. \quad (1)$$

Throughout this manuscript we assume both k_1 and k_2 are strictly positive, otherwise Eq. 1 simplifies to the simple coverage case. $\mathcal{H}$ consists of an additive term accounting for how well the robots cover sites of interest, and a subtractive term penalizing robots staying too close to sites to be avoided. The ratio between factors k_1 and k_2 allows us to rescale the importance of these contrasting goals. The use of max functions indicates that we consider the best coverage robots offer to different points in $\mathcal{E}$, and also the worst covert they experience from points in $\mathcal{E}$. These contributions are weighted by the respective densities, hence it is possible to model different distributions of the various sites. Function $\mathcal{H}$ is to be interpreted as an average accounting for combined coverage and covertness. The problem we wish to study is twofold. First we wish to identify robot locations maximizing Eq. 1, i.e. we aim to solve the following optimization problem:

$$\max_{x_1 \ldots x_n \in \mathcal{E}} \mathcal{H}(x_1, \ldots, x_n). \quad (2)$$

It will soon become evident that while solving this optimization problem one has to be content with a local optimizer. Next we need to define control laws driving the robots to such a point. We conclude this section outlining that in the remaining part of this paper we assume[1] $f(x) = -x^2$.

[1]The choice of a quadratic function is somehow standard in multicenter optimization because it leads to easy to treat derivatives.

Remark: The reader may mistakenly believe Eq. 1 can be reduced to the model given in [8] by introducing a single function accounting for the difference between ϕ_1 and ϕ_2, like $\zeta = k_1\phi_1 - k_2\phi_2$. This is not the case for two reasons. First, the framework introduced in [8] requires a strictly positive density function, whereas a ζ defined as above would in general not satisfy this hypothesis. Moreover, $\max(\phi_1 + \phi_2) \neq \max \phi_1 + \max \phi_2$, hence positiveness cannot be recovered by adding a large constant. These limitations imply that to study the problem considered in this manuscript it is necessary to consider the multicenter function defined in Eq. 1. This inherent difference leads to different critical points and will require a different controller. An additional aspect corroborating the fact that the two models are different is that the model presented in [8] is invariant with respect to singular configurations, whereas Eq. 1 is not invariant (see Section VII for more details about invariance).

IV. Critical Points of the Coverage Function

Let us rewrite Eq. 1 considering the Voronoi partition of $\mathcal{E}$ induced by $x_1, \ldots, x_n$ [5]. Since f was assumed to be non-increasing, then $\mathcal{H}$ can then be rewritten as follows (from now onwards, to ease notation, we omit to explicitly mention the constraint $x_i \in \mathcal{E}$; this requirement will however be explicitly enforced when describing the control law):

$$\mathcal{H}(x_1, \ldots, x_n) = k_1 \sum_{i=1}^{n} \int_{V(x_i)} f(||q - x_i||_2)\phi_1(q)dq - k_2 \sum_{i=1}^{n} \int_{V(x_i)} f(||q - x_i||_2)\phi_2(q)dq. \quad (3)$$

This expression shows that the overall performance is obtained considering $2n$ contributions. The first n terms reward for the coverage offered by the n robots, whilst the last n terms penalizes for lack of covertness. It is important to observe that robot x_i's contribution, either positive or negative, is exclusively determined by its associated Voronoi region $V(x_i)$. To determine the critical points of $\mathcal{H}$ the following theorem, adapted[2] from [7], is needed.

Theorem 1 (Cortès et al., [7]): Let $\mathcal{E}$ be a simple closed convex polygon in $\mathbb{R}^2$, let $x_1, \ldots, x_n$ be n distinct points in $\mathcal{E}$, let $\phi : Q \to \mathbb{R}^+$ be a bounded measurable function on $\mathcal{E}$, and let $f : \mathbb{R}_0^+ \to \mathbb{R}$ be a non-increasing differentiable function. Then, the following function

$$\hat{\mathcal{H}}(x_1, \ldots, x_n) = \int_{\mathcal{E}} \max_{i \in \{1 \ldots n\}} f(||q - x_i||_2)\phi(q)dq \quad (4)$$

is continuously differentiable on every non-singular configuration, and for $i \in \{1 \ldots n\}$:

$$\frac{\partial \hat{\mathcal{H}}}{\partial x_i} = \int_{V(x_i)} \frac{\partial}{\partial x_i} f(||q - x_i||_2)\phi(q)dq. \quad (5)$$

By applying Theorem 1 to Eq. 1 we see that for each non singular configuration the components of the gradient are:

[2]The original theorem considers also the case where f is only piecewise differentiable with finite jump discontinuities. This generalization is not needed here, hence the statement is accordingly simplified.

$$\nabla_i \mathcal{H} = \frac{\partial \mathcal{H}}{\partial x_i} = \int_{V(x_i)} \left[k_1 \frac{\partial}{\partial x_i} f(||q - x_i||_2)\phi_1(q) - k_2 \frac{\partial}{\partial x_i} f(||q - x_i||_2)\phi_2(q) \right] dq. \quad (6)$$

Since we assumed $f(x) = -x^2$, Eq. 6 can be rewritten as follows (see [5], page 116)

$$\frac{\partial \mathcal{H}}{\partial x_i} = 2k_1 \mathrm{A}_{\phi_1}(V(x_i))[\mathrm{CM}_{\phi_1}(V(x_i)) - x_i] - 2k_2 \mathrm{A}_{\phi_2}(V(x_i))[\mathrm{CM}_{\phi_2}(V(x_i)) - x_i]. \quad (7)$$

Critical points of $\mathcal{H}$ occur where every component of the gradient vector is zero, i.e. for each x_i

$$k_1 \mathrm{A}_{\phi_1}(V(x_i))[\mathrm{CM}_{\phi_1}(V(x_i)) - x_i] = k_2 \mathrm{A}_{\phi_2}(V(x_i))[\mathrm{CM}_{\phi_2}(V(x_i)) - x_i].$$

Recalling that $\phi_{1,2}$ were assumed to be strictly positive over $\mathcal{E}$, and $k_1, k_2 > 0$, it follows that both $k_1 \mathrm{A}_{\phi_1}(V(x_i))$ and $k_2 \mathrm{A}_{\phi_2}(V(x_i))$ are strictly positive for every i. Therefore the i-th component is zero if and only if

$$\mathrm{CM}_{\phi_1}(V(x_i)) - x_i = \zeta_i [\mathrm{CM}_{\phi_2}(V(x_i)) - x_i] \quad (8)$$

where the positive constant ζ is defined as

$$\zeta_i = \frac{k_2 A_{\phi_2}(V(x_i))}{k_1 A_{\phi_1}(V(x_i))}. \quad (9)$$

In other words, the two vectors $[\mathrm{CM}_{\phi_1}(V(x_i)) - x_i]$ and $[\mathrm{CM}_{\phi_2}(V(x_i)) - x_i]$ must be parallel and point to the same direction[3]. Finally, it is worth recalling that $\mathcal{H}$ has multiple critical points, and the search for an optimizer has to be intended in a local sense and depends from the initial distribution of the x_is.

A. Remarks

When coverage is sought without considering covertness (i.e. the case of Eq. 4), then Eq. 5 assumes the form:

$$\frac{\partial \hat{\mathcal{H}}}{\partial x_i} = 2\mathrm{A}_{\phi}(V(x_i))[\mathrm{CM}_{\phi}(V(x_i)) - x_i]. \quad (10)$$

Hence, the i-th component of the gradient is zero if and only if robot x_i is located at the center of mass (according to ϕ) of its associated Voronoi cell, i.e. $[\mathrm{CM}_{\phi}(V(x_i)) - x_i] = 0$. Moreover, for each x_i the gradient points in the direction from x_i to $\mathrm{CM}_{\phi}(V(x_i))$. Eq. 7 describes instead a more complex situation that is visually portrayed in Fig. 1.

The gradient points in a direction that compromises between getting closer to $\mathrm{CM}_{\phi_1}(V(x_i))$, i.e. increasing coverage, and farther from $\mathrm{CM}_{\phi_2}(V(x_i))$, i.e. increasing covertness. Factors k_1 and k_2 modulate the relative importance between these two concurrent needs, hence the precise direction is influenced not only by the density distributions ϕ_1 and ϕ_2 but also by the the ratio k_1/k_2. The geometric situation depicted in Fig. 1 not only reconciles our intuition with the analytic results given by Eq. 7, but also gives a visual representation of the condition given by Eq. 8. A critical point is reached when the two vectors $[\mathrm{CM}_{\phi_1}(V(x_i)) - x_i]$ and $[\mathrm{CM}_{\phi_2}(V(x_i)) - x_i]$ are parallel and pointing in the same direction.

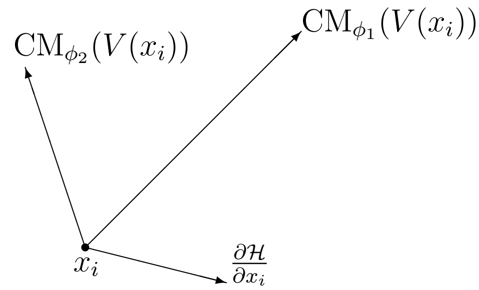


Fig. 1. For the case of observing while maintaining covertness, the gradient for x_i aims to get closer to $\mathrm{CM}_{\phi_1}(V(x_i))$ and farther from $\mathrm{CM}_{\phi_2}(V(x_i))$, hence the resulting vector $\frac{\partial \mathcal{H}}{\partial x_i}$ depicted in the figure.

Finally, the reader should notice that the expression for the gradient of $\mathcal{H}$ (Eq. 6) is distributed in the sense of Voronoi, as defined in [8]. That is to say that in order to compute $\partial \mathcal{H}/\partial x_i$ robot x_i does not need to know the positions of all robots, but only of those contributing to the definition of $V(x_i)$, i.e. those located in a Voronoi region sharing an edge with $V(x_i)$. This observation is fundamental in order to implement a distributed controller for the team.

V. Robot Control

We next consider the problem of designing a distributed robot controller to move the robot team to a configuration that is a local maximum for $\mathcal{H}$. We assume the motion of each robot can be modeled using a first order differential equation, i.e.

$$\dot{x}_i = u_i. \quad (11)$$

An intuitive way to maximize multicenter functions is by imposing that each robot x_i follows the gradient so that the system reaches a stable configuration that locally optimizes $\mathcal{H}$ – a strategy also known as the continuous-time Lloyd algorithm. This approach is at the basis of Bullo et al. work. Unfortunately, this idea cannot be applied to the problem we are considering. In fact, one can prove that while following the gradient of $\hat{\mathcal{H}}$ the robots will never reach a singular configuration nor will they exit the assigned area $\mathcal{E}$ [7]. On the contrary it is simple to determine two functions ϕ_1 and ϕ_2 such that for points on the boundary of $\mathcal{E}$ the gradient of $\mathcal{H}$ points to the outside (see Fig. 2). This complication also follows from the fact that one cannot combine ϕ_1 and ϕ_2 into a single function. The two must be treated separately, and the fact that their difference can be both positive and negative leads to situations where the gradient points outside $\mathcal{E}$.

A more sophisticated control technique is therefore needed. Two eventually equivalent approaches can be fol-

[3]The reader should note that the these vectors point indeed in the same direction, and not in the opposite one, as implied from the fact that $\zeta_i > 0$. Cancellation between these two vectors occur because of the negative sign in the function being optimized.

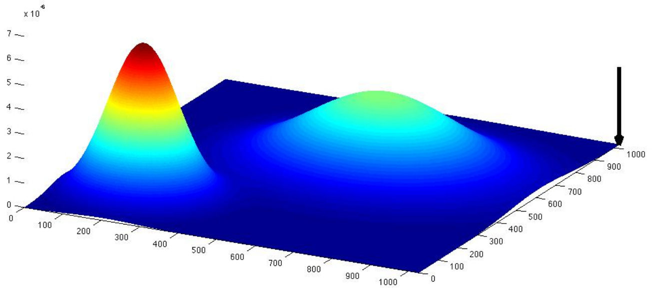


Fig. 2. In the situation depicted in this figure where the higher peak is associated with ϕ_1 and the lower peak is associated with ϕ_2, when the ratio k_2/k_1 is sufficiently large a robot located in the vertex marked by the arrow will be subject to a gradient pointing outside the region.

lowed. The first one casts the problem as a constrained optimization problem and leads to a control strategy that can be implemented following a distributed paradigm. An alternative method could be envisioned by resorting to non-smooth fields [6] to force the robots to remain inside $\mathcal{E}$ once they reach its boundary. Intuitively, if a robot is located on the boundary and its gradient points to the outside, it should not follow its gradient but rather *slide along* the boundary of $\mathcal{E}$ according to the projection of the gradient on it (hence the non-smoothness of the field). This technique however immediately raises the question of how to choose these sliding motions, and how to prove convergence. Therefore we analyze the system as a constrained optimization problem and we will eventually obtain a control strategy that indeed translates to sliding motions on the boundary.

A. Distributed control as distributed constrained optimization

Let us rewrite the optimization problem defined in Eq. 2 as follows:

$$\max_{x_1 \dots x_n} \mathcal{H}(x_1, \dots, x_n)$$
$$\text{subject to } x_i \in \mathcal{E} \quad 1 \le i \le n$$

where each of the n constraints $x_i \in \mathcal{E}$ is equivalent to a set of m linear inequalities involving x_i. That is to say the set of constraints is given by nm linear inequalities:

$$a^j_{i,1} x^1_i + a^j_{i,2} x^2_i \le b^j_i \quad \text{with } 1 \le j \le m, \ 1 \le i \le n.$$

This problem can be solved using the gradient projection method (see [10] for more details; we here follow the notation introduced therein). For sake of completeness we shortly recap the method. Starting from a given configuration $(x_1, \dots, x_n)$ we seek a direction $d = (d_1, \dots, d_n)$ such that $\nabla\mathcal{H} \cdot d > 0$, i.e. by moving along d the value of the objective function increases. d is called feasible direction. As long as none of the x_i is located on the boundary of $\mathcal{E}$ we choose $d = \nabla\mathcal{H}$, and the motion coincides with the formerly mentioned Lloyd algorithm. If one or more robots are located on the boundary of $\mathcal{E}$ then some of the constraints are actually an equality. These equalities are called *active constraints*. Let $\mathbf{A_q}$ be the $k \times 2n$ matrix obtained by stacking the coefficients of the k active constraints, and let $\mathbf{P}$ be the $2n \times 2n$ matrix defined as follows:

$$\mathbf{P} = \mathbf{I} - \mathbf{A_q}^T (\mathbf{A_q} \mathbf{A_q}^T)^{-1} \mathbf{A_q}$$

where $\mathbf{I}$ is the identity matrix of order $2n$. Then, we set $d = \mathbf{P} \cdot \nabla\mathcal{H}^T$. If $d \neq 0$ a feasible direction is found and robot x_i moves along this direction with $u_i = d_i$. On the contrary, if $d = 0$ the Karush-Kuhn-Tucker (KKT) criterion tells if a local maxima was reached. Let

$$\lambda = -(\mathbf{A_q} \mathbf{A_q}^T)^{-1} \mathbf{A_q} \nabla\mathcal{H}^T.$$

If all its components are non positive, then the point is a local maxima and the optimization process terminates. Otherwise one drops from $\mathbf{A_q}$ the row corresponding to the λ_i component most positive and recomputes d with the new $\mathbf{A_q}$.

The essence of the gradient projection method emerges when one or more active constraints are present. In that case by setting $d = \mathbf{P} \cdot \nabla\mathcal{H}^T$ one determines a control where one or more robots *slide* along the boundary of $\mathcal{E}$. This observation reconciles this method with the non-smooth approach we formerly mentioned.

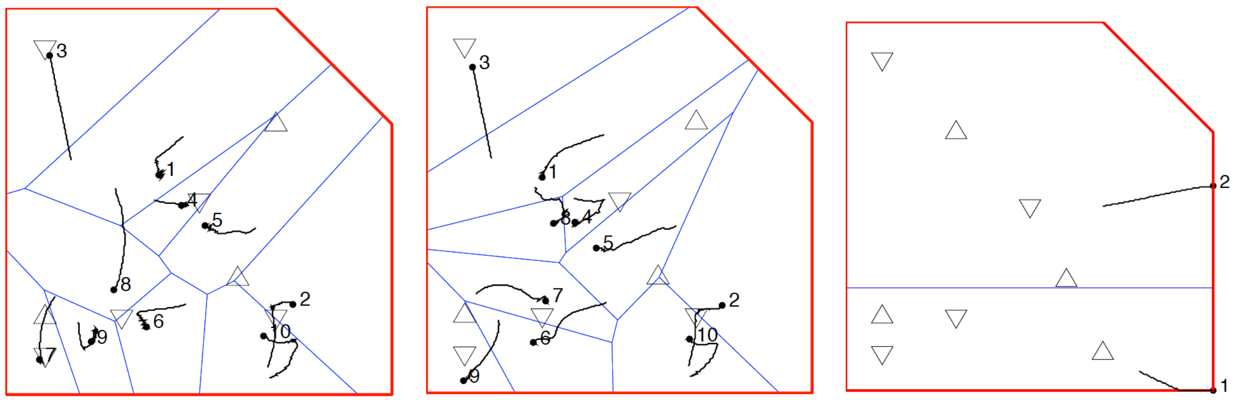
A more careful analysis of the constraints reveals that $\mathbf{P}$ is a block diagonal matrix made of n blocks of dimension 2×2. Let $\mathbf{P_i}$ be these blocks. If robot x_i is not located on the boundary of $\mathcal{E}$ then none of its constraints are active and $\mathbf{P_i}$ is the 2×2 identity matrix. In that situation, $\mathbf{P} \cdot \nabla\mathcal{H}^T$ gives $d_i = \nabla_i\mathcal{H}^T$ and robot x_i follows the gradient. If instead x_i is located on the boundary of $\mathcal{E}$, then one (if it is on an edge) or two (if it is on a vertex) of its linear constraints are active. In that case then d_i is the projection of $\nabla_i\mathcal{H}^T$ along the active constraint(s) and generates the sliding motion. The decomposition of $\mathbf{P}$ into blocks translates into a distributed control strategy. Provided that every robot can autonomously localize itself, then it can determine its set of active constraints and compute its matrix $\mathbf{P_i}$. Given $\mathbf{P_i}$, direction d_i can be computed by using $\nabla_i\mathcal{H}^T$. Therefore the outlined control strategy is also distributed in the sense of Voronoi.

B. Remarks

One should note that the various multicenter problems considered in [5] and related papers are in fact all constrained problems because it is always implied that robots never leave the area of interest. However, because to the assumptions made about the function to be optimized and the density functions considered therein, these constraints do not need to be explicitly taken into account because the gradient flow never pushes the robots outside. These favorable hypotheses however do not hold for the problem at hand, and so the constrained version needs to be explicitly considered.

VI. Simulation Results

We here present the results of some simulations aimed to outline how different performances emerge when changing the ratio between k_1 and k_2 and by varying the number of



(a) $k_1 = 3, k_2 = 2$ (b) $k_1 = 3, k_2 = 6$ (c) $k_1 = 3, k_2 = 8$

Fig. 3. The red contour shows the boundary of the assigned area $\mathcal{E}$, whereas the blue lines show the Voronoi partition induced by the final robot configuration. Black paths display the trajectories followed by the robots while moving towards their final destination. Points of interest are marked by the symbol$\bigtriangledown$, whereas locations to be avoided are marked by the symbol $\bigtriangleup$.

robots[4]. Results are graphically displayed in Fig. 3, and the companion video shows the corresponding evolution over time. Density function ϕ_1 is given by a sum of Gaussians centered on the points marked by the symbol $\bigtriangledown$, whereas ϕ_2 is given by a sum of Gaussians centered on the points marked by the symbol $\bigtriangleup$. In the first two experiments Gaussians in ϕ_1 have the same $\sigma^2 = 0.5$, while Gaussians in ϕ_2 have the same $\sigma^2 = 0.1$. In the last experiment Gaussians in ϕ_2 have instead $\sigma^2 = 0.6$.

Figures 3(a) and 3(b) show how the same initial distributions of robots lead to different final configurations by varying the ratio k_1/k_2. In particular figure 3(b) shows a case with a larger value for k_2 while keeping k_1 constant. It may then be seen that robots 1 and 5 take further cover from the ϕ_2 peak located in the top right of the region, i.e. they still approach the same peak of ϕ_1, but stop at a greater distance. On the bottom left, different trajectories are also produced by robots 6,7, and 9. On the contrary, robots 2,3, and 10 are deployed sufficiently clear of peaks of ϕ_2 and their final trajectories do not show significant variations.

Finally, Fig. 3(c) displays a case where sliding motions occur. Two robots are deployed in $\mathcal{E}$ with a different distributions of peaks for ϕ_1 and ϕ_2, as evidenced by the different layout of the symbols $\bigtriangledown$ and $\bigtriangleup$. Moreover, $k_1 = 1$ and $k_2 = 8$, so that the objective function strongly rewards covertness over coverage. As seen from the trajectories, both robot 1 and robot 2 are pushed towards the boundary of $\mathcal{E}$ while trying to improve the value of $\mathcal{H}$ by gaining covertness. In particular, robot 1 soon approaches the boundary and then slides along the edge until it reaches a vertex where it stops because no feasible direction can be found. Robot 2 also reaches the boundary end then stop because its gradient at that point is orthogonal to the boundary and then its projection is 0.

[4]Matlab© code producing these results is available for download at `http://robotics.ucmerced.edu`. We acknowledge the use of Voronoi routines made available by Andrew Kwok at UC San Diego.

VII. Theoretical Properties

We shortly discuss three theoretical aspects concerning the dynamical system and controller we studied, namely optimality, convergence, and invariance.

Optimality. The function $\mathcal{H}$ being optimized is non convex, continuous and defined on a closed, bounded set.[5] By Weierstrass theorem it then admits minimum and maximum in $\mathcal{E}$. Since we our controller follows the gradient, we will therefore end up (see next discussion about convergence) in a local extrema, but without any guarantee about its global nature. It is easy seeing that determining the global optimum is a NP-hard problem. In fact, if one sets $k_2 = 0$ in Eq. 1, then the original problem studied in [8] is obtained. It is known that finding the optimal solution to this problem is in general NP-hard [1], and therefore by reduction our problem shares the same complexity.

Convergence. Related works in the area of coverage algorithms based on Voronoi tessellation usually establish convergence using LaSalle's principle [5], [17]. However, as evidenced already, our problem is different because deployment takes place in a bounded domain and simply following the gradient for $\mathcal{H}$ may push the agents outside the area. Convergence towards a critical point minimizing $\mathcal{H}$, however, can still be guaranteed even without using LaSalle's principle. In fact, the method described in section V is nothing but the *gradient projection* method known in optimization literature, and this is known to converge to a local optimum (see for example [10], chapter 11, section 4).

Invariance. Invariance is here defined as the property of remaining in non-singular configurations after starting from a non-singular configuration. In different terms, while following the control law, robots should not collide with each when starting from distinct positions. The basic coverage problem enjoys this property, as proven in [7]. As additional evidence of the different nature of the function $\mathcal{H}$ we considered, this is not the case for the problem presented in this paper. In fact, one can easily construct problem

[5]Note that the function is well defined also for *singular* configurations.

instances where two or more robots will eventually move to the same location. This situation can nevertheless be avoided using the approach presented in [15] where a set of linear constraints is included to account for finite robot dimensions. This extension is easily integrated in the framework we propose because it would just add additional constraints that could be accommodated using the gradient projection method formerly illustrated.

VIII. Conclusions

We defined and formalized a novel coverage problem that combines two objectives at once, i.e. providing coverage to a set of known locations and minimizing exposure to a different set of known locations. The problem has been formulated using the theory of multicenter functions, and we have identified fundamental differences with the simpler case considering coverage only. Hence, the proposed control strategy turns out to be more involved and better understood by casting it as an instance of constrained optimization. Various outstanding questions remain for the problem we investigated in this paper. The first one concerns the existence and convergence of distributed discrete time controllers that converge to a local maxima of $\mathcal{H}$. The control law proposed in this manuscript is continuous, but distributed discrete controllers have been shown to exist for the simpler case dealing with coverage only. The existence of distributed discrete time controllers for the problem at hand is still an open problem, also considering that the system follows a radically different flow. Next, considering the *hostile* nature of locations to avoid, it would be interesting to modify the problem formulation to account for the cumulative penalty accrued by being covered while reaching the final configuration. By integrating this penalty over time it is expected different trajectories will emerge, with agents possibly taking detour paths around hostile locations.

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